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STRUCTURAL OPTIMIZATION WITH PIECE-WISE CONCAVE COST FUNCTIONALS

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Abstract—A theorem on a necessary condition for a constrained minimum of piece-wise concave functionals is derived and it is used for obtaining minimum weight solutions for anisotropic cylindrical shells having variable rib depth. Existing methods for plastic optimal design are briefly reviewed.

NOTATION

A, **R**	matrices containing linear operators
A†	pseudo-inverse of **A**
A_N	cross-sectional area per unit width of circumferential stiffeners
A_M	cross-sectional area per unit width of longitudinal stiffeners
b	aggregate width of stiffeners per unit width of shell
B	Banach space
$c_1, c_2, \mathbf{d}_k, d_{k0}, k$	known constants
d	depth of stiffeners
D	domain of integration
$\bar{D}$	subset of D
$f()$	function or functional
h	length of R_M-type region
H	Hilbert space
I	index set
$\mathbf{L}_i, \mathbf{L}_j$	vectors consisting of linear operators
m	positive integer
M	longitudinal bending moment in shells
M_0	yield moment in simple bending
M_θ	circumferential moment
M_r	radial moment
$p()$	load
p_M	load risisted by moments
p_N	load resisted by hoop forces
q_i	slack functions
R^n, R^m	real Euclidean space
R_N, R_M	basic feasible regions (BFR's)
r	radial coordinate
R	radius of middle surface of shell
s_i	Lagrangian multipliers
$t_{\min}, t_0, t_{\max}$	limiting values of plate thickness
V	total volume of stiffeners (ribs)
$\mathbf{x}$	unspecified vector function
x	longitudinal coordinate for cylindrical shell
$\mathbf{x}_0$	optimal solution

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X_k	closed set in a Banach space
y_l, y_j	specified scalar functions
β	number of crease equations
γ	number of constraint equations
δ	number of constraint inequalities
λ, λ_i	real numbers
ν	non-dimensional parameter $=\sqrt{\frac{2q}{b\sigma_y}}$
Φ	total cost
Ψ	specific cost function
σ_y	yield stress
Γ	feasible set

INTRODUCTION

Concave programming is a technique [1, 2] for finding the global minimum of a piecewise concave,† or briefly *PC*, function defined on a set in Euclidean space. In this paper, the principle of concave programming is extended from Euclidean space to function-spaces. A scalar valued function defined on a set in a function space is called a *functional*. The value of a functional depends on the choice of one or more functions termed "independent" functions. A *PC* function defined on a set in function space is termed a *PC functional.*

Minima of *PC* functions defined on a set in Euclidean space cannot be located by standard methods of differential calculus because, at their minima, they are non-differentiable with respect to any variation. In a geometrical sense, the slope of the graph of a *PC* function is discontinuous in all directions at its lowest point. Similarly, a *PC* functional is non-differentiable along its minima with respect to any variation of the independent functions. Hence classical variational methods (Euler equations) can be used only after a suitable reformulation involving Lagrangian multipliers [20] and slack functions. The proposed approach eliminates the necessity for this reformulation, results in a simpler procedure and gives a much smaller number of possible solutions than the Lagrangian multiplier method.

PC functionals are associated with a wide range of optimization problems. As an example, consider the problem of optimizing a rigid–perfectly plastic axisymmetric plate, Fig. 1(a), having a Tresca yield condition, Fig. 1(b), and a plate thickness constrained to a specified range of values $(t_{\min}, t_{\max})$. Let the price per unit volume of plate material be c_1 for thicknesses $t \leq t_0$ and c_2 for $t \geq t_0$. Then the total cost can be shown to be

$$\Phi = \int_D \Psi(M_\theta, M_r) r \, \mathrm{d}r \tag{1}$$

where $\Psi(M_\theta, M_r)$ is defined by the following equations, see Fig. 1(c),

$$\left.\begin{array}{l} \Psi(M_0) = c_1 t_{\min} \quad \text{for } k|M_0|^{\frac{1}{2}} \leq t_{\min} \quad \text{and} \\ \Psi(M_0) = c_i k|M_0|^{\frac{1}{2}} \quad \text{for } k|M_0|^{\frac{1}{2}} \geq t_{\min}, \\ k|M_0|^{\frac{1}{2}} \leq t_{\max}, \qquad M_0 = |M_\theta| + |M_r| + |M_\theta - M_r|, \\ i = 1 \quad \text{for } k|M_0|^{\frac{1}{2}} \leq t_0, \qquad i = 2 \quad \text{for } k|M_0|^{\frac{1}{2}} \geq t_0 \end{array}\right\}, \tag{2}$$

† For definition of *PC* functions see Appendix.

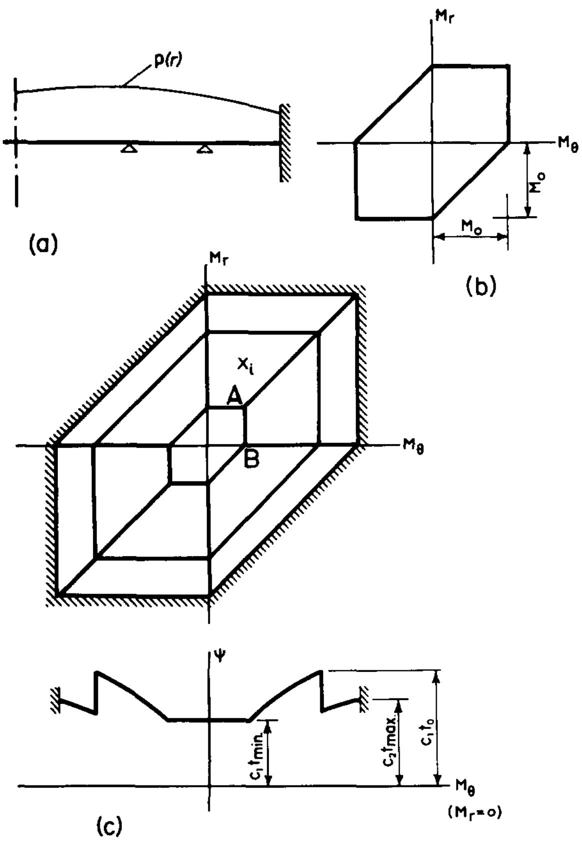

FIG. 1. Axisymmetric Tresca-plate and the corresponding specific width functions.

M_θ and M_r are the radial and circumferential moments, r is the radial coordinate and k is a constant. The total cost Φ is to be minimized subject to the equilibrium condition

$$(M_r.r)'' - M_\theta' = r.p(r) \tag{3}$$

where $p(r)$ is the external load and primes denote differentiation with respect to r.

It will be shown subsequently that a piece-wise continuous optimal solution can only contain regions in which either $k|M_\theta|^{\frac{1}{2}}$ or $k|M_r|^{\frac{1}{2}}$ or $k|M_\theta - M_r|^{\frac{1}{2}}$ equals t_{min}, t_0 or t_{max} or $M_\theta = 0$ or $M_r = 0$ or $M_\theta = M_r$. Along such a solution, the functional $\Phi(M_\theta, M_r)$ is, in general, non-differentiable with respect to any variation of (M_θ, M_r) that satisfies (3).

Several methods are available for the plastic optimal design of structures associated with convex and piece-wise differentiable specific cost functions. Drucker and Shield [3] introduced the uniform energy dissipation method for minimizing a certain class of convex cost functionals. Marcal *et al.* [4, 5] developed a more general theory for the plastic optimal design of structures associated with convex cost functions. Rozvany and Charrett [6] extended the Prager–Shield theory to multiple loading conditions and multi-component systems.

Marcal [7] derived optimal solutions for Tresca sandwich plates associated with bilinear specific cost functions. Megarefs [8, 9] and Rozvany and Melchers [10, 11] used statical methods for deriving optimal solutions for axially symmetric Tresca sandwich-plates and reinforced slabs, respectively. Sheu and Prager [12] optimized sandwich-plates having piece-wise constant thickness.

However, relatively little is known about solutions involving non-convex specific cost functions. Hopkins and Prager [13], Freiberger and Tekinalp [14], Onat *et al.* [15] and Mroz [16] derived solutions representing local minima for the volume of circular and annular solid plates. However, it was pointed out by Brotchie [17] and in greater detail by Kozlowski and Mroz [18] that the absolute minimum volume of solid plates is theoretically zero since the solution reduces to ribs of infinite depth. Frames associated with non-convex cost functions were discussed by Megarefs and Hodge [19].

Definitions and proofs are given in the Appendix. The main part of the paper deals with the practical aspects of these methods with a particular reference to plastic optimal design.

OPTIMIZATION PROBLEMS ASSOCIATED WITH A PIECE-WISE CONCAVE FUNCTIONAL

Consider the following class of problems:

$$\text{Minimize} \quad \Phi = \int_D \Psi(\mathbf{x})\,\mathrm{d}D \tag{4}$$

$$\text{subject to} \quad \mathbf{L}_l\mathbf{x} = y_l \qquad (l = 1, 2, \ldots, \gamma) \tag{5}$$

$$\text{and} \quad \mathbf{L}_j\mathbf{x} \geq y_j \qquad (j = 1, 2, \ldots, \delta) \tag{6}$$

where Φ is the cost functional defined on a set contained in a Hilbert space H;

D is a set contained in m-dimensional Euclidean space R^m; $\mathbf{x} = (x_1, \ldots, x_n) \in H$ where $x_1, \ldots, x_n$ are unspecified functions defined on D;

y_l and y_j are specified functions contained in the range of $\mathbf{L}_l$ and $\mathbf{L}_j$, respectively;

$\mathbf{L}_l = (L_{1l}, \ldots, L_{nl})$ and $\mathbf{L}_j = (L_{1j}, \ldots, L_{nj})$ where L_i and L_{ij} $(i = 1, 2, \ldots, n)$ are linear operators having a domain and a closed range in Hilbert spaces;

$\Psi(\mathbf{x})$, called the *specific cost function*, is a *PC* function defined on R^n and having only linear *crease equations* [1, 2]

$$\mathbf{d}_k\mathbf{x} = d_{k0} \qquad (k = 1, 2, \ldots, \beta) \tag{7}$$

where $\mathbf{d}_k$ and d_{k0} are constants. This means [1, 2] that R^n can be divided into sets X_i $(i = 1, 2, \ldots, \alpha)$ such that $\Psi(\mathbf{x})$ is concave on any one X_i and any X_i is the intersection of half-spaces in R^n, see Fig. 1(c). Equations (7) define boundary hyperplanes of such half-spaces.

Equations (5) are called *constraint equations* and inequalities (6) are called *constraint inequalities*. The corresponding equations

$$\mathbf{L}_j\mathbf{x} = y_j \qquad ((j = 1, 2, \ldots, \delta) \tag{8}$$

are called *modified constraints*. Both crease equations and modified constraints are termed *conditional equations*. The function $\mathbf{x}_0$ along which Φ takes on its global constrained minimum constitute an *optimal solution.* Any function $\mathbf{x} \in H$ satisfying (5) and (6) is said to be *feasible.*

The theorem that follows is restricted to a class of problems for which a piece-wise continuous optimal solution exists. The proof of the following theorem is outlined in the Appendix: *it is a necessary condition for at least one optimal solution* $\mathbf{x}_0$ *that the domain D can be divided into subsets* D_h $(h = 1, 2, \ldots, \varepsilon)$ *such that a linearly independent system of n conditional and constraint equations is satisfied by* $\mathbf{x}_0$ *at all interior points of any one such subset (Theorem* 1).

If a function $\mathbf{x}^*$ is feasible and satisfies the requirements of Theorem 1 then D_h $(h = 1, 2, \ldots, \varepsilon)$ are called *basic feasible regions* or BFR's. It follows that $\mathbf{x}^*$ satisfies the same $(n-\gamma)$ conditional equations in any one BFR.

If the specific cost function is strictly *PC* then all optimal solutions consist of BFR's. If it is *PC* but not strictly *PC* and the set of all optimal solutions is bounded, then at least one optimal solution consists of BFR's.

It is to be remarked that the foregoing theorem is proved for linear inequality constraints and linear crease-equations. The same theorem can be extended easily to nonlinear inequality constraints and crease-equations by using the same considerations as in concave programming [1, 2].

Returning to the *example* discussed in the introduction, the number of unspecified functions is $n = 2$ and the number of constraint equations is $\gamma = 1$ [equation (3)]. The conditional equations are:

$$\left.\begin{array}{llll} \pm k^2 M_i = t_{\min}^2, & \pm k^2 M_i = t_{\max}^2, & \pm k^2 M_i = t_0^2 & (i = \theta, r) \\ \pm k^2(M_\theta - M_r) = t_{\min}^2, & \pm k^2(M_\theta - M_r) = t_{\max}^2, & \pm k^2(M_\theta - M_r) = t_0^2 & \\ M_\theta = 0, \quad M_r = 0, & M_\theta = M_r & & \end{array}\right\}. \tag{9}$$

By theorem (1), $n-\gamma = 2-1 = 1$ of the above twenty-one equations must be satisfied in each BFR of the plate.

In order to reduce the number of possible solutions admitted by the foregoing necessary condition (Theorem 1), each crease-equation in equation (9) can be supplemented with inequalities defining the line segment along which the crease equation is valid. The crease equation $k^2 M_\theta = t_{\min}^2$, for example, is valid only in between points A and B in Fig. 1(c). Hence the inequalities $0 \leq k^2 M_r \leq t_{\min}^2$ could be added to this crease-equation.

In the special case of $t_{\min} = 0$, $t_0 = 0$ and $t_{\max} = \infty$, the only BFR's are

$$\text{(i) } M_\theta = 0 \qquad \text{(ii) } M_r = 0 \qquad \text{(iii) } M_\theta = M_r. \tag{10}$$

The same necessary condition for a local minimum was obtained by Mroz [16] who only considered the foregoing special case and used a different method. Unfortunately, the condition $t_{\max} = \infty$ results in solutions that are not piece-wise continuous and hence Theorem 1 cannot be used for deriving optimal solutions for this special case.

Naturally, similar BFR's can be obtained for non-axisymmetric plates by first assuming that the directions of optimal principal moments are fixed and then repeating the above procedure in the corresponding curvilinear system.

Once the BFR's are determined, the only unknown quantities are the topology and shape of the region boundaries.

The proposed technique is demonstrated in detail by examples of plastic optimal design involving anisotropic cylindrical shells stiffened by very densely spaced ribs in two directions. If the aggregate width of the ribs per unit width of shell is not preassigned then the optimal volume is again theoretically zero and the solution consists of densely spaced, infinitely deep ribs. However, by fixing the aggregate width of the ribs per unit width, piece-wise continuous optimal solutions can be obtained.

EXAMPLE

Plastic optimal design of anisotropic cylindrical shells

Consider a class of cylindrical shells [Fig. 2(a)] in which the width (W) and spacing (S) of ribs have preassigned values in both longitudinal and circumferential directions but the

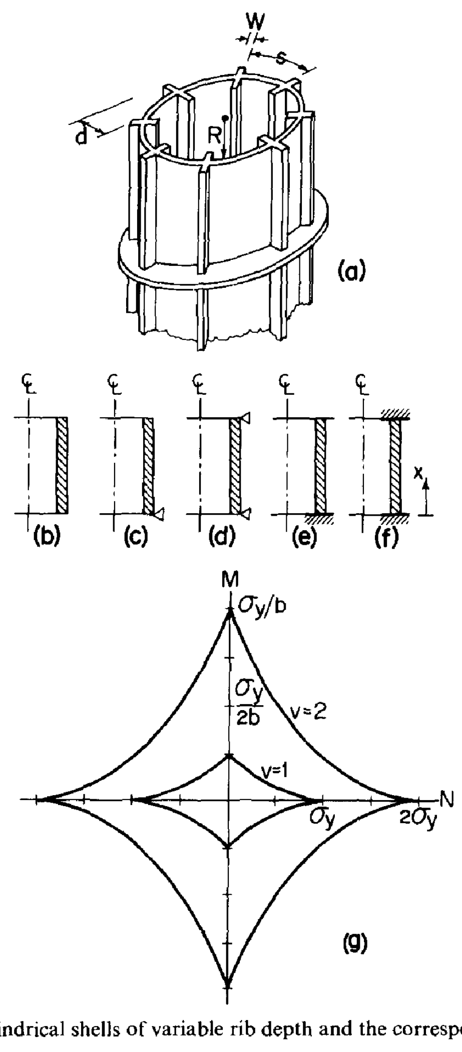

FIG. 2. Anisotropic cylindrical shells of variable rib depth and the corresponding cost function.

depth (d) of the ribs is a variable quantity. It is assumed that: (i) the entire load is resisted by the ribs; (ii) there is no interaction between the yield capacity of longitudinal and circumferential ribs; (iii) the spacing (S) of the ribs is small in comparison with the radius (R) of the shell;† (iv) both the loading and the boundary conditions are rotationally symmetric.

A non-uniform internal pressure $p(x)$ will be considered. The equilibrium equation for this problem is

$$\frac{\mathrm{d}M}{\mathrm{d}x}+\frac{N}{R} = p_M+p_N = p \tag{11}$$

where M is the longitudinal bending moment per unit width;
N is the hoop force per unit width;
R is the radius of the middle surface of the shell;
x is the longitudinal coordinate [Fig. 2(f)];
p_M and p_N are the loads resisted by the bending moments and hoop forces, respectively.

The cross-sectional area A_N of the circumferential ribs per unit width is given by

$$A_N = \frac{|N|}{\sigma_y} \tag{12}$$

where σ_y is the yield stress in simple tension. The plastic section modulus of the ribs being $bd^2/4$, the yield moment capacity of the longitudinal ribs per unit width is

$$|M| = \frac{bd^2}{4}\sigma_y \tag{13}$$

and their area A_M is

$$A_M = bd \tag{14}$$

where b is the aggregate width of the longitudinal ribs over a unit shell width and d is the depth of the longitudinal ribs. From (13) and (14),

$$A_M = 2\left|\sqrt{\left(\frac{b}{\sigma_y}\right)}.\sqrt{M}\right|. \tag{15}$$

The total volume of the ribs is

$$V = 2\pi R\int v\,\mathrm{d}x = 2\pi R\int_0^L (A_N+A_M)\,\mathrm{d}x = 2\pi R\int_0^L \left(\left(\frac{|N|}{\sigma_y}\right) + \left(2\left|\sqrt{\left(\frac{b}{\sigma_y}\right)}.\sqrt{M}\right|\right)\right)\mathrm{d}x \tag{16}$$

where L is the length of the shell.

† In Fig. 2(a), a simplified representation is used insofar as the spacing (S) of the ribs is quite large in comparison to the radius (R). In the actual problem considered, the ratio S/R is infinitesimal, but the quantity W/S has a finite, pre-assigned value.

Introducing the notation $\bar{V} = V\sigma_y/2\pi R$ and $k = 2|\sqrt{b\sigma_y}|$, where $\bar{V}$ is called the "modified volume", equation (16) can be rewritten

$$\bar{V} = \int_0^L (|N| + k|\sqrt{M}|)\,\mathrm{d}x. \tag{16a}$$

The specific cost function [Fig. 2(g)] is clearly piece-wise concave (see Definition 1 in the Appendix) and the crease (conditional) equations are

$$N = 0 \tag{17}$$

and

$$M = 0. \tag{18}$$

Since the number of unspecified functions in the integrand is two (N and M), Theorem 1 implies that two constraint and conditional equations must be satisfied in each BFR.

The constraint equation to be satisfied is the equilibrium equation (11) and the conditional equation is either (17) or (18). Hence the solution must consist of two types of regions:

$$R_M\text{-type BFR with } N = 0\ (p_N = 0,\ p_M = p)$$

$$R_N\text{-type BFR with } M = 0\ (p_M = 0,\ p_N = p).$$

Next, it will be shown that *an R_M type BFR must be adjacent to an external support in an optimal solution for the class of shells considered* (*Proposition* 1). The proof is by contraposition.

Assume that: (i) in the optimal solution $N \equiv 0$ over the interior of region ($b \le x \le a$) and $M \equiv 0$ in the adjacent regions; (ii) no external support occurs in the same interval [Fig. 3(a)]. Assuming some non-zero internal pressure throughout the region *ab*, it follows from the equilibrium condition (11) that M takes on some non-zero value over the same region but M can only become zero in the adjacent regions if concentrated hoop forces A and B act along the boundaries of the region *ab*. Further, it follows from equilibrium that

$$A + B = \int_b^a p(x)\,\mathrm{d}x. \tag{19}$$

Then equations (16) and (19) show that the volume V_N of the circumferential ribs takes on the same value in the region ($b \le x \le a$) both in the case of $M \equiv 0$ (R_N-type BFR)

$$[V_N]_b^a = \int_b^a \frac{|p(x)\,\mathrm{d}x|}{\sigma_y} \tag{20}$$

and in the case of $N \equiv 0$ (R_M-type BFR)

$$[V_N]_b^a = \frac{|A| + |B|}{\sigma_y}. \tag{21}$$

Whilst the former solution represents zero cross-sectional area for the longitudinal ribs in region *ab*, the latter solution would require *additional* volume for the longitudinal ribs [Fig. 3(a)] and hence it cannot be an optimal solution (QED). This shows that R_M-type

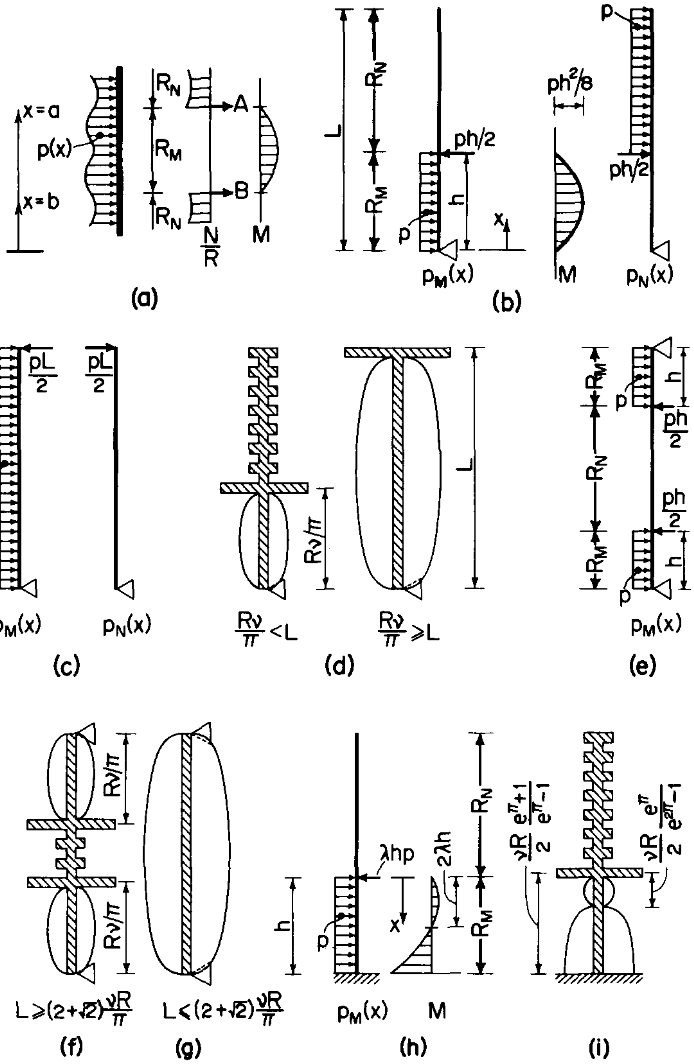

FIG. 3. Minimum volume solution for anisotropic cylindrical shells.

regions must have at least one of their boundaries along an external support. It follows that the optimal solution may only have the following topologies:

(a) R_N-type BFR throughout the shell;
(b) R_M-type BFR throughout the shell;
(c) R_M-type BFR adjacent to the support(s) and R_N-type BFR over the remainder of the shell.

In the case of type (c) topology, the only remaining unknown quantities are:

(a) the width of the R_M-type BFR's;
(b) the magnitude of the concentrated hoop force (if any) along the boundary of the R_M-type region.

The optimal solutions will be determined for the boundary conditions shown in Figs. 2(b)–(g), considering the special case of a uniformly distributed internal pressure p = const.

(i) *Unsupported cylindrical shells* [*Fig.* 2(*b*)]. If the shell has no external lateral supports, the solution is trivial since Proposition 1 implies that the solution consists of a single R_N-type region. From equation (16a) the modified volume of the ribs then becomes

$$\bar{V} = RpL. \tag{22}$$

(ii) *Cylindrical shell having simple support at one end* [*Fig.* 2(*c*)]. Assuming topology (c) first, the load components p_M and p_N and the moment field M are shown in Fig. 3(b). The moment field is given by the equation

$$M = p\left[\frac{hx}{2} - \frac{x^2}{2}\right] \qquad \text{for } 0 \le x \le h \tag{23}$$

and the modified volume, from equation (16a) is:

$$\begin{aligned}\bar{V} &= \left(L - \frac{h}{2}\right)Rp + k\int_0^h \sqrt{\left(p\left[\frac{hx}{2} - \frac{x^2}{2}\right]\right)}\,\mathrm{d}x \\ &= \left(L - \frac{h}{2}\right)Rp + \left[k\sqrt{\left(\frac{p}{2}\right)}\frac{h^2}{8}\arcsin\frac{x - h/2}{h/2}\right]_0^h \\ &= \left(L - \frac{h}{2}\right)Rp + k\sqrt{\left(\frac{p}{2}\right)}\frac{h^2\pi}{8}.\end{aligned} \tag{24}$$

For the optimal value of h,

$$\frac{\mathrm{d}\bar{V}}{\mathrm{d}h} = -\frac{Rp}{2} + \frac{h\pi}{4}k\sqrt{\left(\frac{p}{2}\right)} = 0 \tag{25}$$

which gives

$$h_{\text{opt}} = \frac{R}{\pi}\sqrt{\left(\frac{2p}{b\sigma_y}\right)} = \frac{Rv}{\pi} \tag{26}$$

where $v = \sqrt{(2p/b\sigma_y)}$ is a non-dimensional quantity. The optimal volume for topology (b) is derived by substituting (26) into the final expression under (24):

$$\bar{V} = Rp\left(L - \frac{Rv}{4\pi}\right). \tag{27}$$

Clearly, the modified volume in equation (27) is smaller than the one given by topology (a) [see equation (22)].

Naturally, equation (27) is valid only if $h_{\text{opt}} \le L$. For $h_{\text{opt}} \ge L$, topology (b) becomes the optimal solution and the modified volume can be calculated by substituting L for h in equation (24):

$$\bar{V} = \frac{L}{2}Rp + k\sqrt{\left(\frac{p}{2}\right)}\frac{L^2\pi}{8} = Rp\left(\frac{L}{2} + \frac{L^2\pi}{4Rv}\right). \tag{28}$$

For $h_{\text{opt}} = Rv/\pi = L$, both (27) and (28) give

$$\bar{V} = \tfrac{3}{4}RLp. \tag{29}$$

The cross sections of the two types of optimal solutions are shown in Fig. 3(d).

(iii) *Cylindrical shell having simple supports at both ends [Fig. 2(d)].* For this boundary condition, both topologies (b) and (c) will be considered. In the case of topology (c) there is an R_M-type region at both ends [Fig. 3(e)]. The optimal value of h can be shown to be the same as for the previous boundary condition, see equation (26). The modified volume for this boundary condition and topology is

$$\bar{V} = Rp\left(L - \frac{Rv}{2\pi}\right). \tag{30}$$

If topology (b) is adopted, the modified volume is given by the second term in the right hand side of equation (28):

$$\bar{V} = k\sqrt{\left(\frac{p}{2}\right)}\frac{L^2\pi}{8} = \frac{pL^2\pi}{4v}. \tag{31}$$

By equating the right hand sides of equations (30) and (31), the limiting case of these topologies can be determined:

$$L = (2+\sqrt{2})\frac{vR}{\pi} = 1{\cdot}086778\frac{vR}{\pi}. \tag{32}$$

If L is greater and smaller, respectively, than the value in equation (32), topology (c) and topology (b) gives the optimal solution, [see Figs. 3(f) and (g)]. It is interesting to note that if the length of the shell is decreased progressively, the optimal solution changes suddenly from topology (c) [Fig. 3(f)] to topology (b) well before the two R_M-type regions reach one another.

(iv) *Cylindrical shell with one clamped end and one free end [Fig. 2(e)].* Considering topology (c), the moment field will be given by [Fig. 3(h)]

$$M = \lambda hpx - \frac{px^2}{2}. \tag{33}$$

The corresponding modified volume from equation (16a) is

$$\bar{V} = [L-(1-\lambda)h]pR + k\left[\int_0^{2\lambda h}\sqrt{\left(\lambda hpx - \frac{px^2}{2}\right)}\,\mathrm{d}x + \int_{2\lambda h}^{h}\sqrt{\left(\frac{px^2}{2} - hpx\right)}\,\mathrm{d}x\right]. \tag{34}$$

Equation (34) yields:

$$\frac{\bar{V}}{pR} = L-(1-\lambda)h + \frac{h^2}{vR}\left[\lambda^2\pi - \lambda^2\log\frac{\sqrt{(1-2\lambda)}+1-\lambda}{\lambda} + (1-\lambda)\sqrt{(1-2\lambda)}\right]. \tag{35}$$

Then the condition $\partial\bar{V}/\partial h = 0$ gives

$$h = \frac{(1-\lambda)vR}{2F(\lambda)} \tag{36}$$

where $F(\lambda)$ is the expression in square brackets in equation (35). Substituting the right hand side of (36) for h in (35), we get

$$\frac{\bar{V}}{pR} = L - \frac{(1-\lambda)^2 vR}{4F(\lambda)}. \tag{37}$$

Applying the condition $\partial \bar{V}/\partial \lambda = 0$ to equation (37), we get

$$\lambda = \frac{2\,e^{\pi}}{(e^{\pi}+1)^2} = 0{\cdot}0794158 \tag{38}$$

or

$$\lambda = 0.$$

Substituting the first value of λ into equations (36) and (37),

$$h_0 = \frac{vR}{2}\frac{e^{\pi}+1}{e^{\pi}-1} = 0{\cdot}545166\,vR \tag{39}$$

$$\frac{\bar{V}}{pR} = L - \frac{vR}{4}\frac{e^{2\pi}+1}{e^{2\pi}-1} = L - 0{\cdot}250935vR \tag{40}$$

where h_0 is the optimal value of h. $\lambda = 0$ gives

$$\frac{\bar{V}}{pR} = L - \frac{vR}{4} \tag{41}$$

which is greater than the volume in equation (40). Hence the optimal values of h and λ are given by equations (38) and (39). The optimal value of λh is

$$\lambda h_0 = \frac{e^{\pi}}{e^{2\pi}-1}Rv = 0{\cdot}0432948\,Rv. \tag{42}$$

The optimal solution is shown in Fig. 3(i).

The foregoing optimal solution is valid only if $L \leq h_0$. For $L \geq h_0$, topology "B1" shown in Figs. 4(a) and (b) gives the optimal solution and the volume becomes

$$\frac{\bar{V}}{pR} = \lambda L + \frac{L^2}{vR}F(\lambda). \tag{43}$$

For a stationary value of $\bar{V}$,

$$\frac{1}{pRL}\frac{\partial \bar{V}}{\partial \lambda} = 1 + \frac{L}{vR}F'(\lambda) = 0 \tag{44}$$

which gives

$$F'(\lambda) = 2\lambda\left[\pi - \log\frac{1-\lambda+\sqrt{(1-2\lambda)}}{\lambda} - 2\sqrt{(1-2\lambda)}\right] = \frac{vR}{L}. \tag{45}$$

Since an explicit solution for equation (45) has not been found, the optimal value of λ was determined for various values of vR/L from equation (45) on a computer and substituting this value of λ into equation (43), the optimal volumes were calculated. Topology "B1" corresponds to curve segment AB in Fig. 5.

However, at a value of $vR/L = 0{\cdot}49531$ which corresponds to $\lambda_{opt} = 0{\cdot}0186$, the optimal solution suddenly changes to topology "B2" shown in Figs. 4(c) and (d). The value of λ_{opt} is discontinuous at this point since it changes from the above value to zero. The volume for topology "B2" is given by

$$\bar{V} = \sqrt{\left(\frac{p}{2}\right)}\int_0^L k\sqrt{x^2}\,dx = \sqrt{\left(\frac{p}{2}\right)}k\frac{L^2}{2} \tag{46}$$

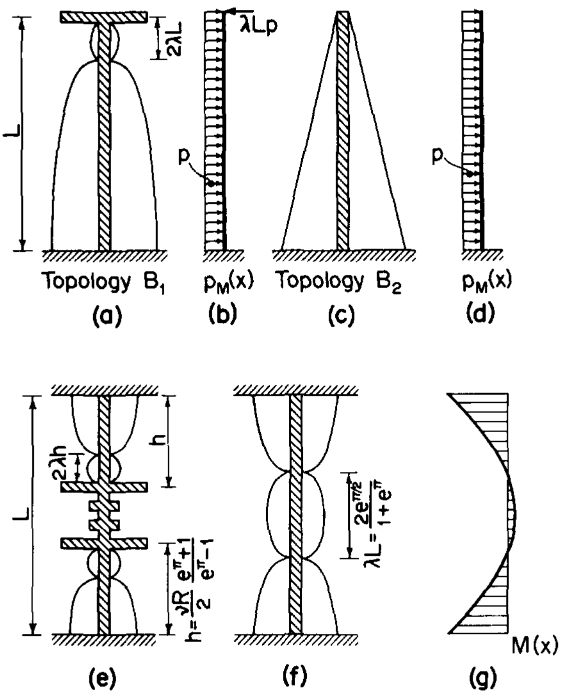

FIG. 4. Minimum volume solution for anisotropic cylindrical shells.

which reduces to

$$\frac{\bar{V}}{p} = \frac{L}{\nu}. \tag{47}$$

(v) *Cylindrical shell having both ends clamped* [*Fig.* 2(*g*)]. In this case, two topologies must be considered. For topology (c), the solution contains an R_M-type region at both ends and

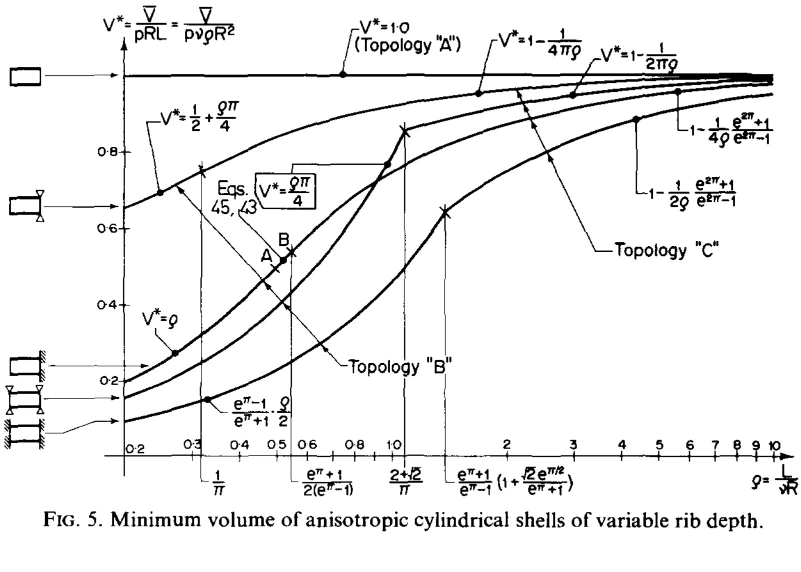

FIG. 5. Minimum volume of anisotropic cylindrical shells of variable rib depth.

the optimal values of λ and h are the same as for shells clamped at one end only [Fig. 4(e)]. The optimal volume for this topology is

$$\frac{\bar{V}}{pR} = L - \frac{\nu R}{2}\frac{e^{2\pi}+1}{e^{2\pi}-1}. \tag{48}$$

If topology (b) is adopted [Fig. 4(f)] and L is the length of the positive part of the moment field [Fig. 4(g)], the moment field over one half of the shell is given by

$$M_x = p\left(\frac{\lambda^2 L^2}{8} - \frac{x^2}{2}\right). \tag{49}$$

The total volume of the ribs is

$$\bar{V} = 2k\left(\int_0^{L/2} \sqrt{\left(\frac{p\lambda^2 L^2}{8} - \frac{px^2}{2}\right)}\,dx + \int_{\lambda L/2}^{L/2} \sqrt{\left(\frac{px^2}{2} - \frac{p\lambda^2 L^2}{8}\right)}\,dx\right) \tag{50}$$

which yields

$$\frac{\bar{V}}{p} = \frac{L^2}{2\nu}\left(\frac{\pi\lambda^2}{2} + \sqrt{(1-\lambda^2)} - \lambda^2 \log\frac{1+\sqrt{(1-\lambda^2)}}{\lambda}\right). \tag{51}$$

The condition $\partial\bar{V}/\partial\lambda = 0$ gives

$$\lambda = \frac{2\,e^{\pi/2}}{1+e^{\pi}} = 0{\cdot}398537. \tag{52}$$

Substituting this optimal value of λ into equation (51), we get

$$\frac{\bar{V}}{p} = \frac{L^2}{2\nu}\frac{e^{\pi}-1}{e^{\pi}+1} = 0{\cdot}458576\frac{L^2}{\nu}. \tag{53}$$

The range of validity of this solution can be obtained by equating the optimal volumes for topology (c) [equation (48)] and topology (b) [equation (53)] and expressing L in terms of R:

$$\bar{L} = \frac{e^{\pi}+1}{e^{\pi}-1}R\nu\left(1 + \frac{\sqrt{(2)}\,e^{\pi/2}}{e^{\pi}+1}\right) = 1.397595\,R\nu. \tag{54}$$

For $L \leq \bar{L}$ topology (c) gives the optimal solution and for $\bar{L} \leq L$ topology (b) is optimal.

The modified volumes for cylindrical shells with various boundary conditions are compared in Fig. 5. The optimality of the solutions presented was confirmed by numerical calculations. It will be seen that the optimal solutions are somewhat similar to the ones obtained by Shield [21] for cylindrical sandwich shells. Optimal solutions for solid cylindrical shells were obtained by Onat and Prager [22] and Freiberger [23], for reinforced shells by Mroz [24] and for cylindrical sandwich shells obeying the von Mises yield condition by Freiberger [25].

CONCLUDING REMARKS

A technique for finding the global minimum of piece-wise concave functionals has been outlined. In general, the foregoing problem is not amenable to usual variational methods and hence the Euler equations are replaced by so-called "crease equations" and/or "modified constraints". The proposed technique was demonstrated on optimization problems involving the plastic design of anisotropic cylindrical shells.

The authors have developed an alternative method for handling non-convex functionals which is based on kinetic considerations derived by a variational approach [6]. The kinematic requirements for the above problem are:

$$\varepsilon = \operatorname{sgn} N, \qquad \kappa = k \operatorname{sgn} M/2\sqrt{|M|} \tag{55}$$

where ε is the hoop strain and κ is the longitudinal curvature. The kinematic method has confirmed the results of this paper but will be discussed elsewhere.

Equation (55) gives a constant rate of plastic dissipation along exterior surfaces of ribs. This is consistent with the principle of uniform energy dissipation [3] and with a more recent optimality criterion for elastic structures by Masur [26]. The solutions in the foregoing problem are essentially determinate and hence they are valid for both plastic and elastic structures.

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APPENDIX

Definitions and a proof of Theorem 1

(i) *Definition* 1. A real valued function(al) $f(\,)$ defined on a closed set X_k contained in a Banach space B is said to be *concave, convex, strictly concave, strictly convex* and *linear*, respectively, if for any finite set of points $\mathbf{x}_i \in X_k (i = 1, 2, \ldots, m)$ and for any finite set of positive numbers $\lambda_i (i = 1, 2, \ldots, m)$ with $\sum_{i=1}^{m} \lambda_i = 1$ and $(\sum_{i=1}^{m} \lambda_i \mathbf{x}_i) \in X_k$,

$$f_k\left(\sum_{i=1}^{m} \lambda_i \mathbf{x}_i\right) \{\geq, \leq, >, <, =\} \sum_{i=1}^{m} \lambda_i f_k(\mathbf{x}_k) \tag{56}$$

where $f_k(\mathbf{x}) = \lim_{\mathbf{y}\to\mathbf{x}} f(\mathbf{y})$ with $\mathbf{y} \in \text{int } X_k$ and the symbol "int" refers to the interior of a set.

(ii) *Definition* 2. A real valued function(al) $f(\,)$ defined on a closed set $X \subseteq B$ is said to be *piece-wise concave, piece-wise strictly concave* and *piece-wise linear* or briefly *PC, PSC* and *PL*, respectively, if there exists a finite collection of sets $X_i (i \in I)$ such that
(a) the sets int $X_i (i \in I)$ are disjoint and X is the union of all $X_i (i \in I)$;
(b) $f(\,)$ is concave, strictly concave and linear on any one $X_i (i \in I)$.

(iii) *Lemma* 1. Let $\mathbf{L}_r$ $(r = 1, 2, \ldots, q)$ be vectors consisting of n linear operators having a domain and a closed range contained in Hilbert spaces, $\mathbf{x} = (x_1, \ldots, x_n)$ an unspecified vectorfunction and y_r $(r = 1, 2, \ldots, q)$ specified scalarfunctions.

If the system $\mathbf{L}_r\mathbf{x} = y_r$ $(r = 1, 2, \ldots, q)$, $q < n$ has one solution $\mathbf{z}$, then there exists a vectorfunction $\mathbf{d} \neq 0$ such that for all real λ, $\mathbf{z} + \lambda\mathbf{d}$ is a solution.

(iv) *Proof.* After rearranging, $\mathbf{A}\bar{\mathbf{x}} = y_r - \mathbf{R}\bar{\bar{\mathbf{x}}}$ where $\bar{\mathbf{x}} = (x_1, \ldots, x_q)$, $\bar{\bar{\mathbf{x}}} = (x_{q+1}, \ldots, x_n)$ and $\mathbf{A}$ and $\mathbf{R}$ are matrices containing operators associated with $\bar{\mathbf{x}}$ and $\bar{\bar{\mathbf{x}}}$, respectively. If the system has a solution $(\bar{\mathbf{x}}_0|\bar{\bar{\mathbf{x}}}_0)$ then $\mathbf{A}$ has at least a pseudo-inverse $\mathbf{A}^\dagger$ which is a linear operator, see Ref. [19, pp. 163–165]; hence $(\bar{\mathbf{x}} + \lambda\mathbf{A}^\dagger\mathbf{R}\mathbf{k}|\bar{\bar{\mathbf{x}}} + \nu\mathbf{k})$ is a solution for any element $\mathbf{k}$ of the domain of $\mathbf{R}$ (QED).

(v) *Definition* 3. *The feasible set* $\Gamma \subseteq H$ consists of all vector-functions $\mathbf{x}$ defined on D that satisfy all constraint equations and constraint inequalities.

(vi) *Definition* 4. If $\mathbf{d}_k\mathbf{x} = \mathrm{d}_{k0}$ is a crease equation then

$$\mathbf{d}_k\mathbf{x} \{\leq, \geq\} d_{k0} \tag{57}$$

are called *crease inequalities*.

(vii) *Proof of Theorem* 1. The theorem is proved for piece-wise strictly concave specific cost functions. The proof is by contraposition.

Let $\mathbf{z} \in \Gamma$ satisfy less than n linearly independent constraint and conditional equations on the interior of some set $\bar{D} \subseteq$ of finite Lebesgue measure. Divide $\bar{D}$ into subsets $\bar{D}_h$ $(h \in I)$

such that the same conditional equations and crease inequalities are satisfied on the interior of any one $\bar{D}_h$. Lemma 1 implies that given a subset $\bar{D}_h$ there exists a vectorfunction $\mathbf{d} \neq 0$ defined on D_h such that for all real λ, $\mathbf{z}+\lambda\mathbf{d}$ satisfies all constraint and conditional equations that are satisfied by $\mathbf{z}$ on $\bar{D}_h$.

$\mathbf{z}$ also satisfies *in a strict sense* the same set of crease inequalities and constraint inequalities throughout int $\bar{D}_h$. This implies that if $\max_{\bar{D}_h} |\lambda\mathbf{d}|$ is chosen smaller than some specified $\varepsilon > 0$ then $\mathbf{z}+\lambda\mathbf{d}$ will also satisfy such inequalities at all points of int $\bar{D}_h$.

It will be recalled that the specific cost function $\psi(\ )$ is strictly concave on any one set $X_i \subseteq R^h$ and that X_i is the intersection of halfspaces which are the graphs of crease inequalities. The conclusion of the last two paragraphs imply that at any interior point of $\bar{D}_h$,

$$\mathbf{z} \in X_i, \qquad z+\lambda\mathbf{d} \in X_i, \qquad z-\lambda\mathbf{d} \in X_i \tag{58}$$

for some X_i and

$$\mathbf{z} \in \Gamma, \qquad \mathbf{z}+\lambda\mathbf{d} \in \Gamma, \qquad \mathbf{z}-\lambda\mathbf{d} \in \Gamma \tag{59}$$

where $\lambda\mathbf{d} \neq 0$. Then by the strict concavity property of $\psi(\ )$ on X_i, adopting $\lambda_1 = \lambda_2 = \frac{1}{2}$,

$$\psi[\mathbf{z}(\mathbf{p})] > \tfrac{1}{2}\psi[\mathbf{z}(\mathbf{p})+\lambda\mathbf{d}(\mathbf{p})]+\tfrac{1}{2}\psi[\mathbf{z}(\mathbf{p})-\lambda\mathbf{d}(\mathbf{p})] \tag{60}$$

for all $\mathbf{p} \in \text{int}\, \bar{D}_h$ $(h \in I)$. Integrating both sides of (58) on $\bar{D}$ and noting that $\bar{D} = \bigcup \bar{D}_h$ but the boundaries of $\bar{D}_h - s$ have a zero Lebesgue measure,

$$\int_{\bar{D}} \psi(\mathbf{z})\,\mathrm{d}D > \frac{1}{2}\int_{\bar{D}} \psi(\mathbf{z}+\lambda\mathbf{d})\,\mathrm{d}D + \frac{1}{2}\int_{\bar{D}} \psi(\mathbf{z}-\lambda\mathbf{d})\,\mathrm{d}D \tag{61}$$

which implies that Φ is smaller along either $\mathbf{z}+\lambda\mathbf{d}$ or $\mathbf{z}-\lambda\mathbf{d}$ than along $\mathbf{z}$. Since all those three vectorfunctions are in the feasible set Γ, Φ cannot take on its constrained global minimum along $\mathbf{z}$ (QED).

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Абстракт—Выводится теорема необходимого условия ограниченного минимума функционалов кусочно вогнутых. Она используется для получения решений для минимума веса для анизотропных, цилиндрических оболоуек, обладающих переменной высотой ребер. Дается краткий обзор существующих мотодов оптимального расчета в пластической области.